Some Notes on the Inverse Domination Conjecture

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Note.

Graphs in this talk are simple.

Definition

- For a vertex \( v \in V(G) \), \( N(v) = \{ u | \{ u, v \} \in E(G) \} \) and \( N[v] = \{ v \} \cup N(v) \). If \( S \subset V(G) \), \( N[S] = \bigcup_{v \in S} N[v] \). A set \( D \) is dominating if \( N[D] = V(G) \).
- The domination number of \( G \), denoted \( \gamma(G) \), is the smallest cardinality of a set that dominates \( V(G) \).

\( \gamma(G) = 3 \) in the graph below.
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- The \textit{inverse domination number} of $G$, denoted $\gamma'(G)$, is the smallest cardinality of an inverse dominating set for some minimum dominating set.

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Assertion
Kulli and Sigarkanti introduced the inverse domination number and made the following assertion:

$$\gamma'(G) \leq \alpha(G)$$

Initial Proof
Their proof of the assertion was incorrect. This was noticed by Gayla Domke, Jean Dunbar, Teresa Haynes, Steve Hedetniemi, and Lisa Markus.

Inverse Domination Conjecture
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Inverse Domination Conjecture
Given a graph \( G \) with no isolated vertices,

\[ \gamma'(G') \leq \alpha(G'). \]
True Fact (Johnson, Prier, and Walsh)

If $G$ has no isolated vertices and $\gamma(G) \leq 4$ then $\gamma'(G) \leq \alpha(G)$.

The Problem

However, their method is difficult to extend to $\gamma(G) \geq 5$. 
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Definition

A fixed inverse domination number of $G$, denoted $\Gamma'_D(G)$, is the smallest cardinality of an inverse dominating set for a fixed minimum dominating set $D$.

Observation

Clearly $\gamma(G) \leq \gamma'(G) \leq \Gamma'_D(G)$.

Conjecture

If $G$ has no isolated vertices then $\Gamma'_D(G) \leq \alpha(G)$ for any minimum dominating set $D$. 
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Conjecture

If $G$ has no isolated vertices then $\Gamma'_D(G) \leq \alpha(G)$ for any minimum dominating set $D$. 
Based on the observation that $\gamma'(G) \leq \Gamma'_D(G)$, our conjecture is certainly a strengthening of the inverse domination conjecture.

Also, note that for $G = K_{2,m}$, $\gamma'(G) = 2$, but there exists a minimum dominating set $D$ such that $\Gamma'_D(G) = m$.

However, in this extremal case and a few others we’ve studied, $\Gamma'_D(G) \leq \alpha(G)$
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Sub-Conjecture

If $G$ has no isolated vertices and $\gamma(G) = 5$ then $\Gamma'_D(G) \leq \alpha(G)$ for any minimum dominating set $D$.

Let $D$ be a minimum dominating set of $G$ and let $I$ be a maximum independent set where $I \cap D$ is as small as possible. For illustration, suppose $|I \cap D| = 3$.

Goal: Form an inverse dominating set using $I \setminus D$ or reach some contradiction.
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Goal: Form an inverse dominating set using $I \setminus D$ or reach some contradiction.
Suppose both $d_0$ and $d_1$ are adjacent to $I \setminus D$. 

\begin{center}
\begin{tikzpicture}
\draw (0,0) circle (2cm) node {$I$};
\draw (4,0) circle (2cm) node {$D$};
\draw (1,0) -- (3,0);
\draw (2,1) -- (2,-1);
\draw (2,0) node {$d_0$};
\draw (2,1) node {$d_1$};
\draw (2,-1) node {$d_2$};
\draw (3.5,0) node {$d_3$};
\draw (1.5,0) node {$d_4$};
\end{tikzpicture}
\end{center}
What is dominating the neighborhood of $I \cap D$ in $V(G) \setminus (I \cap D)$?
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Answer: $I \setminus D$
A lot of case analysis is used, but the idea is that if this were not the case, either $|I|$ would be bigger, $|I \cap D|$ would be smaller, or we could form an inverse dominating set of the proper size.
Let $X$ be a set of no more than $|I \cap D|$ vertices dominating each vertex in $I \cap D$

So $(I \cup X) \setminus D$ is an inverse dominating set where

$\Gamma'_D(G) \leq \alpha(G)$.
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So $(I \cup X) \setminus D$ is an inverse dominating set where $\Gamma'_D(G) \leq \alpha(G)$. 
Suppose both $d_0$ and $d_1$ are not adjacent to $I \setminus D$.

Then $I \setminus D$ is not dominated by $D$ ... a contradiction!
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Then $I \setminus D$ is not dominated by $D$ ... a contradiction!
Suppose only $d_1$ is adjacent to $I \setminus D$. (We say $D'$ is the set of vertices not adjacent to $I \setminus D$; so, $|D'| = 1$.)
Let $Y = V(G) \setminus D$

Let $Z$ be a maximal independent set in $Y$ that contains at least one vertex that is adjacent to a vertex in $I \cap D$ and then is adjacent to as many vertices in $D \setminus I$ as possible.
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Let \( Z \) be a maximal independent set in \( Y \) that contains at least one vertex that is adjacent to a vertex in \( I \cap D \) and then is adjacent to as many vertices in \( D \setminus I \) as possible.
Let $U$ be the set of at most $|D \setminus (D' \cup (I \cap D))|$ vertices in $V(G) \setminus D$ adjacent to all of the vertices in $D \setminus (D' \cup (I \cap D))$.

Let $X$ be the set of at most $|(I \cap D) \setminus D_0|$ vertices in $V(G) \setminus D$ adjacent to every vertex in $(I \cap D) \setminus D_0$. 
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![Diagram](image)
If $|D'| = 0$ or if $|D'| = 2$
then $\Gamma'D(G) \leq \alpha(G)$ as before.

If $|Z| \leq |(I \setminus D) \cup D_0| - 1$
then $Z \cup X \cup U$ is an inverse dominating set of an appropriate size.

If $|Z| > |(I \setminus D) \cup D_0|$ then $Z \cup ((I \cap D) \setminus D_0)$ is a larger independent set than $I$.

If $|Z| = |(I \setminus D) \cup D_0|$ then $Z \cup ((I \cap D) \setminus D_0)$ is an independent set with smaller intersection to $D$. 
If \(|D'| = 0\) or if \(|D'| = 2\) then \(\Gamma_D'(G) \leq \alpha(G')\) as before.

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Exemplary Case

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In Conclusion

- There are similar arguments for the cases when $|I \cap D| = \{1, 2\}$; however, some oddities occur.
- The benefit of this approach is that if sub-conjecture is not true, it would point to a possible counterexample.
- This approach is also a plausible approach in that Drs. Peter Johnson and Jessica McDonald (Auburn University) have used this technique to re-prove the inverse domination conjecture for claw-free graphs.
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Thank You For Your Kind Attention!

Figure: War Eagle!