Some Notes on the Inverse Domination Conjecture

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Definition Theory

Note.

Graphs in this talk are simple.

Definition

- For a vertex $v \in V(G)$, $N(v) = \{u \mid \{u, v\} \in E(G)\}$ and $N[v] = \{v\} \cup N(v)$. If $S \subset V(G)$, $N[S] = \cup_{v \in S} N[v]$. A set D is *dominating* if $N[D] = V(G)$.
- The *domination number* of G, denoted $\gamma(G)$, is the smallest cardinality of a set that dominates $V(G)$.

$\gamma(G) = 3$ in the graph below.

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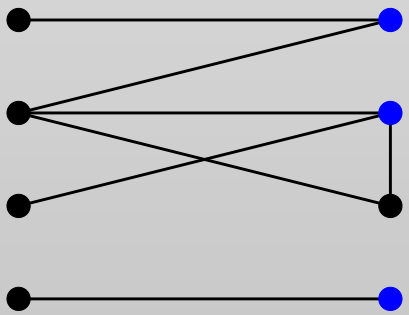

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- Given a minimum dominating set D of a graph G with no isolated vertices, an *inverse dominating set* is a set of vertices in $V(G) \setminus D$ that dominates G.
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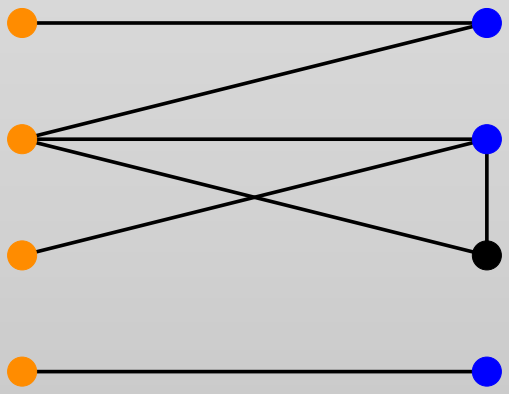

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$\gamma(G) \leq \gamma'(G)$

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Initial Conjecture

Assertion

Kulli and Sigarkanti introduced the inverse domination number and made the following assertion:

$$\gamma'(G) \leq \alpha(G)$$

Initial Proof

Their proof of the assertion was incorrect. This was noticed by Gayla Domke, Jean Dunbar, Teresa Haynes, Steve Hedetniemi, and Lisa Markus.

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True Fact (Johnson, Prier, and Walsh)

If G has no isolated vertices and $\gamma(G) \leq 4$ then $\gamma'(G) \leq \alpha(G)$.

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However, their method is difficult to extend to $\gamma(G) \geq 5$.

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Strengthening the Conjecture

Definition

A *fixed inverse domination number* of G, denoted $\Gamma'_D(G)$, is the smallest cardinality of an inverse dominating set for a fixed minimum dominating set D.

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Clearly $\gamma(G) \leq \gamma'(G) \leq \Gamma'_D(G)$.

Conjecture

If G has no isolated vertices then $\Gamma'_D(G) \leq \alpha(G)$ for any minimum dominating set D.

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- Based on the observation that $\gamma'(G) \leq \Gamma'_D(G)$, our conjecture is certainly a strengthening of the inverse domination conjecture.
- Also, note that for $G = K_{2,m}$, $\gamma'(G) = 2$, but there exists a minimum dominating set D such that $\Gamma'_D(G) = m$.
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Baby Steps

Sub-Conjecture

If G has no isolated vertices and $\gamma(G) = 5$ then $\Gamma'_D(G) \leq \alpha(G)$ for any minimum dominating set D.

Let D be a minimum dominating set of G and let I be a maximum independent set where $I \cap D$ is as small as possible. For illustration, suppose $|I \cap D| = 3$

Goal: Form an inverse dominating set using $I \setminus D$ or reach some contradiction.

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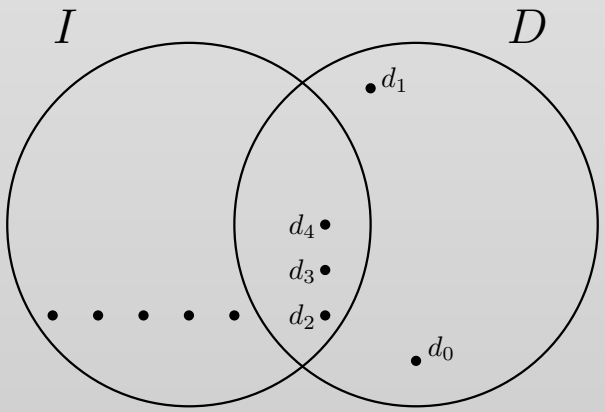

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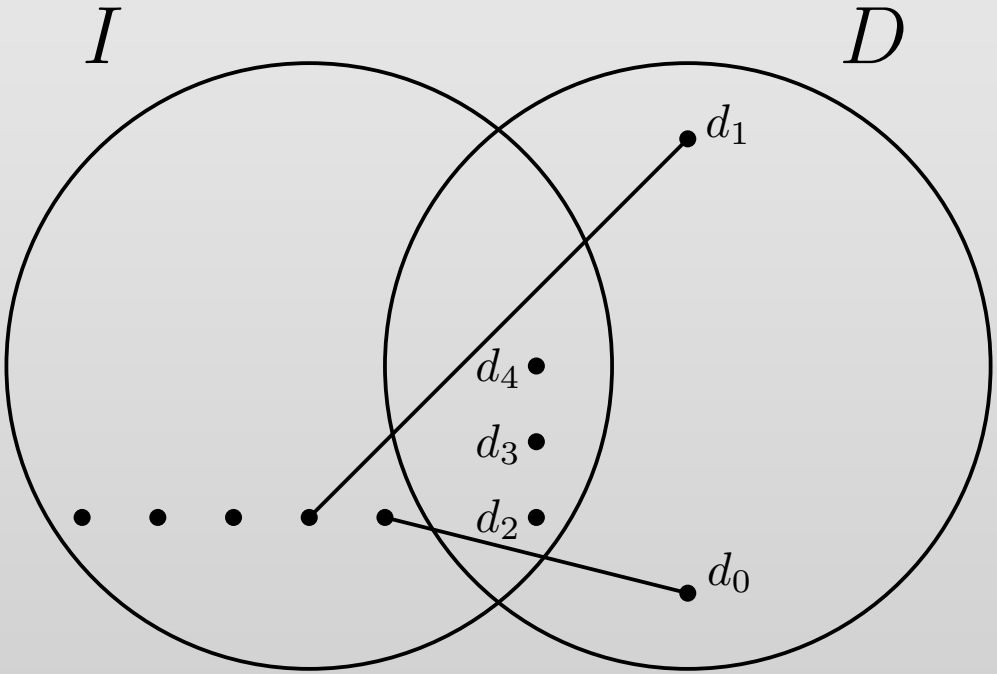
Exemplary Case

Suppose both d_0 and d_1 are adjacent to $I \setminus D$.



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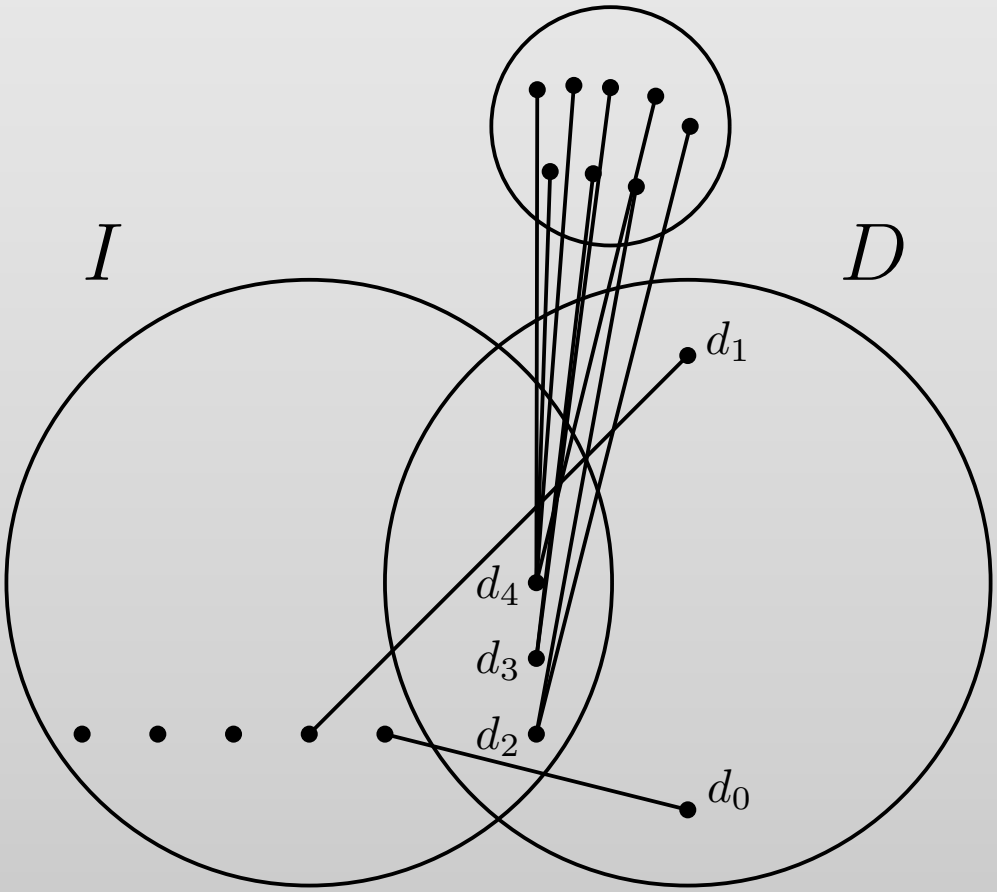
What is dominating the neighborhood of $I \cap D$ in $V(G) \setminus (I \cap D)$?



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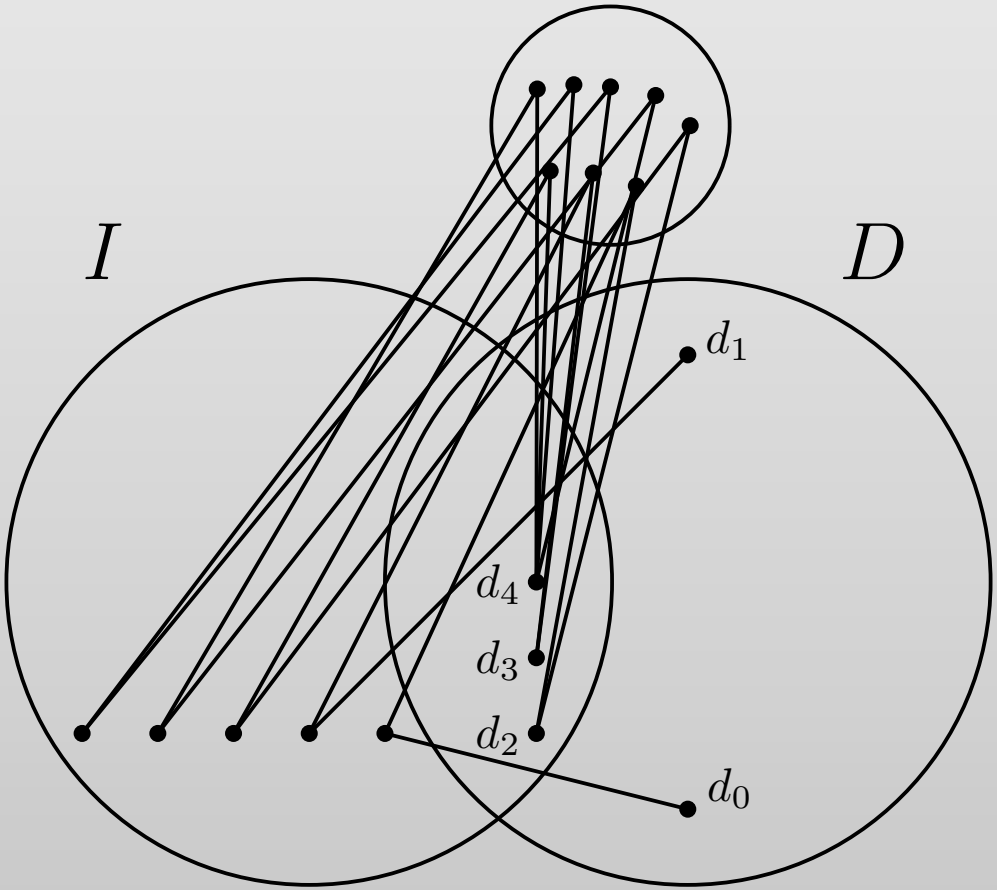

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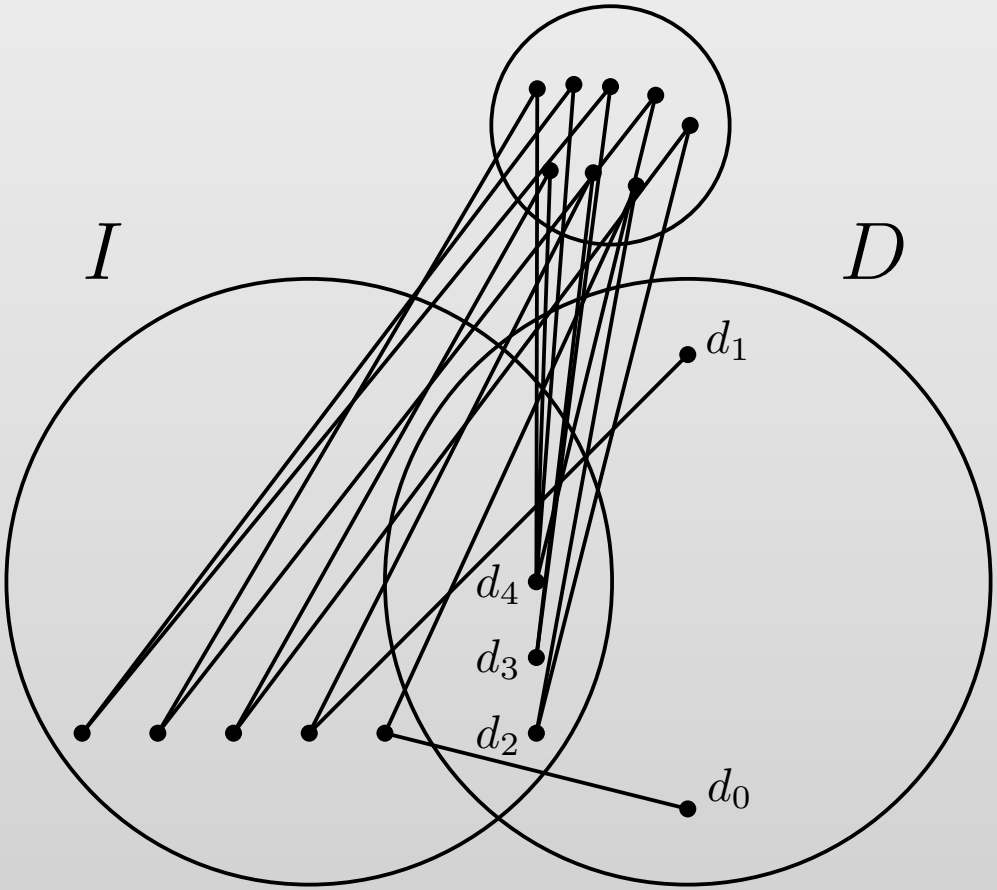
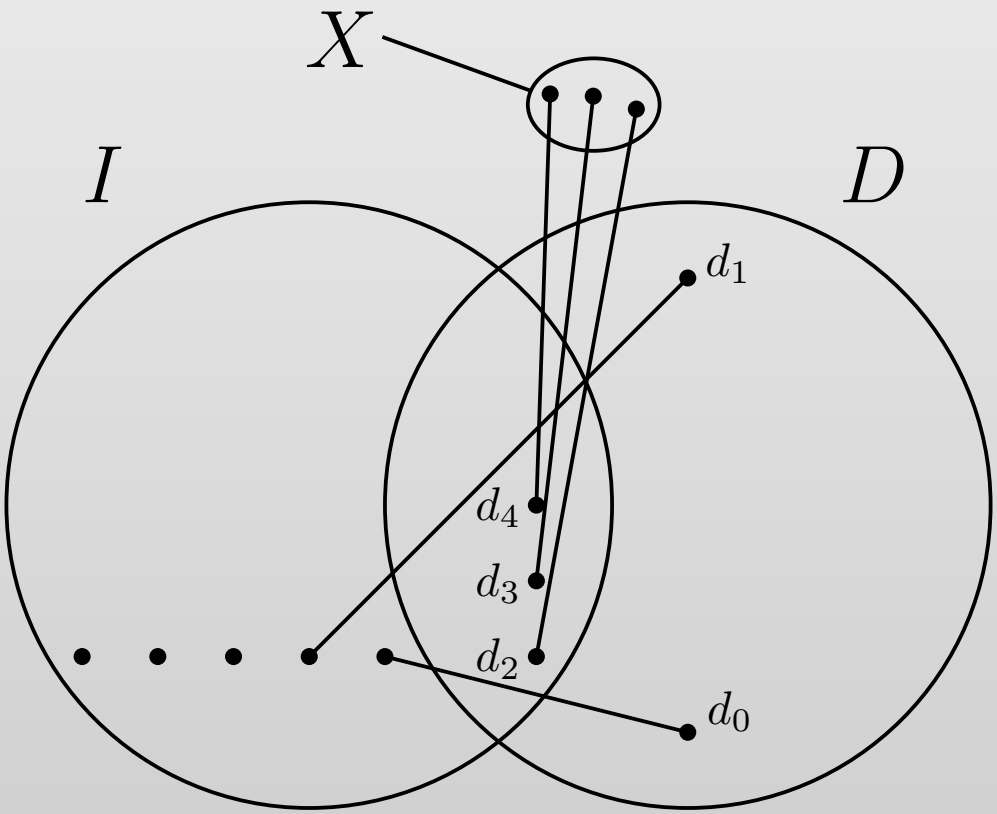


A lot of case analysis is used, but the idea is that if this were not the case, either $|I|$ would be bigger, $|I \cap D|$ would be smaller, or we could form an inverse dominating set of the proper size.

Exemplary Case

Let X be a set of no more than $|I \cap D|$ vertices dominating each vertex in $I \cap D$



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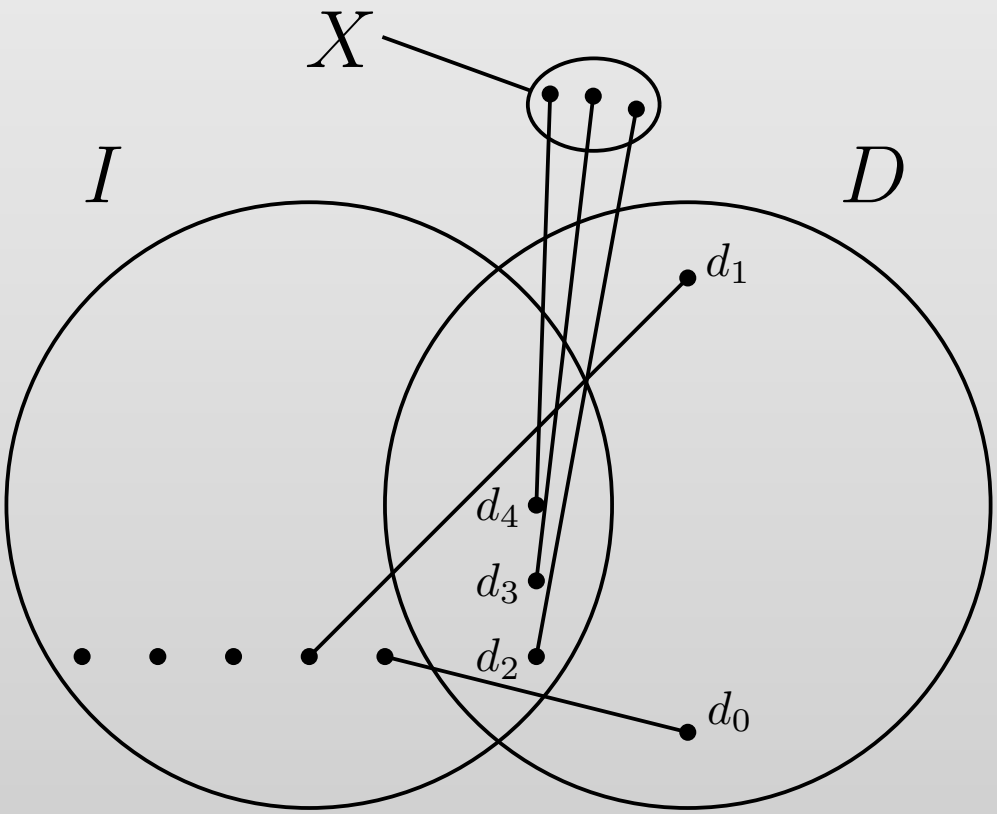

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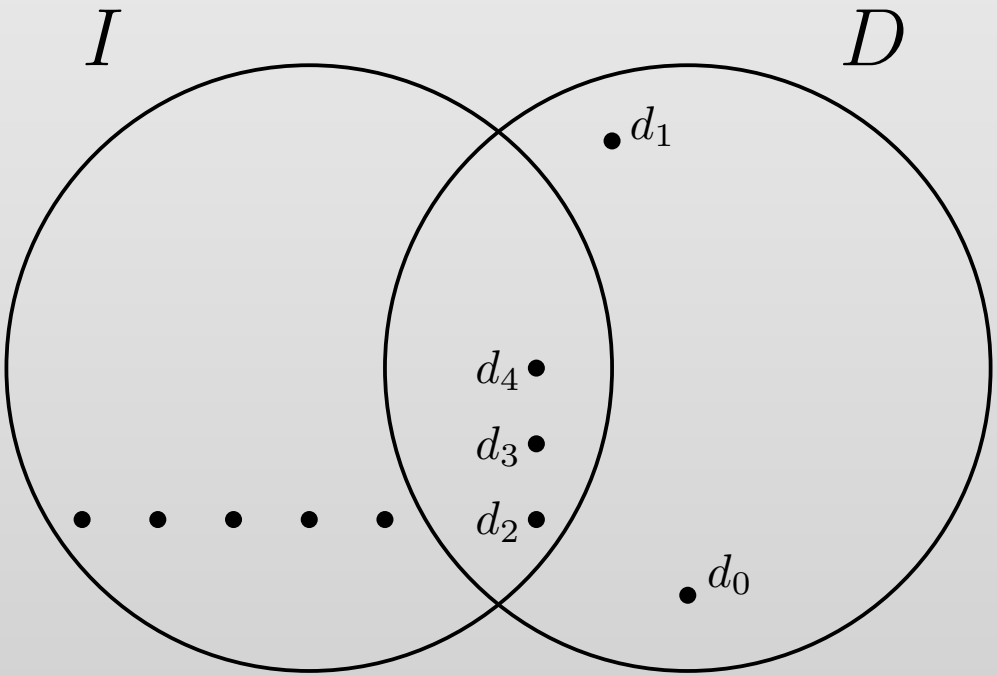

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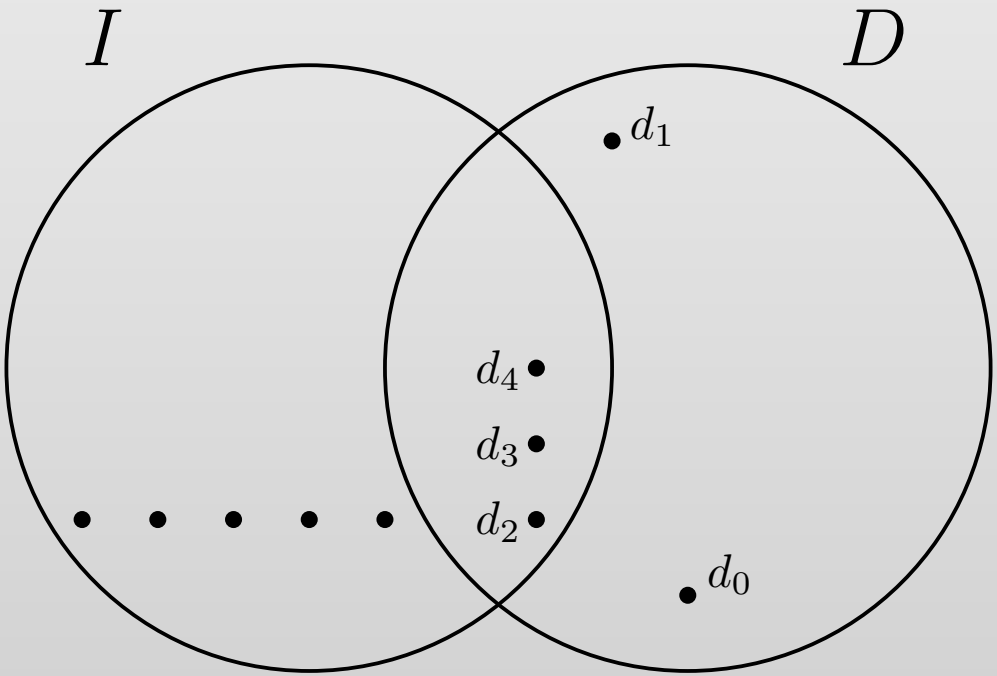
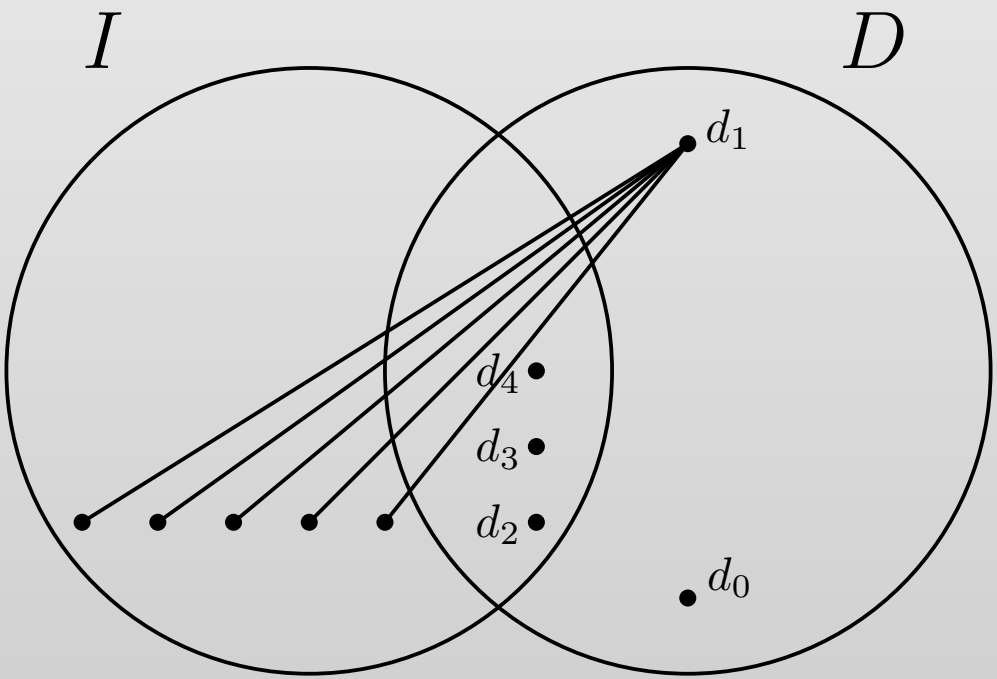


Then $I \setminus D$ is not dominated by $D \ldots$ a contradiction!

Exemplary Case

Suppose only d_1 is adjacent to $I \setminus D$. (We say D' is the set of vertices not adjacent to $I \setminus D$; so, $|D'| = 1$.)



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- Let $Y = V(G) \setminus D$
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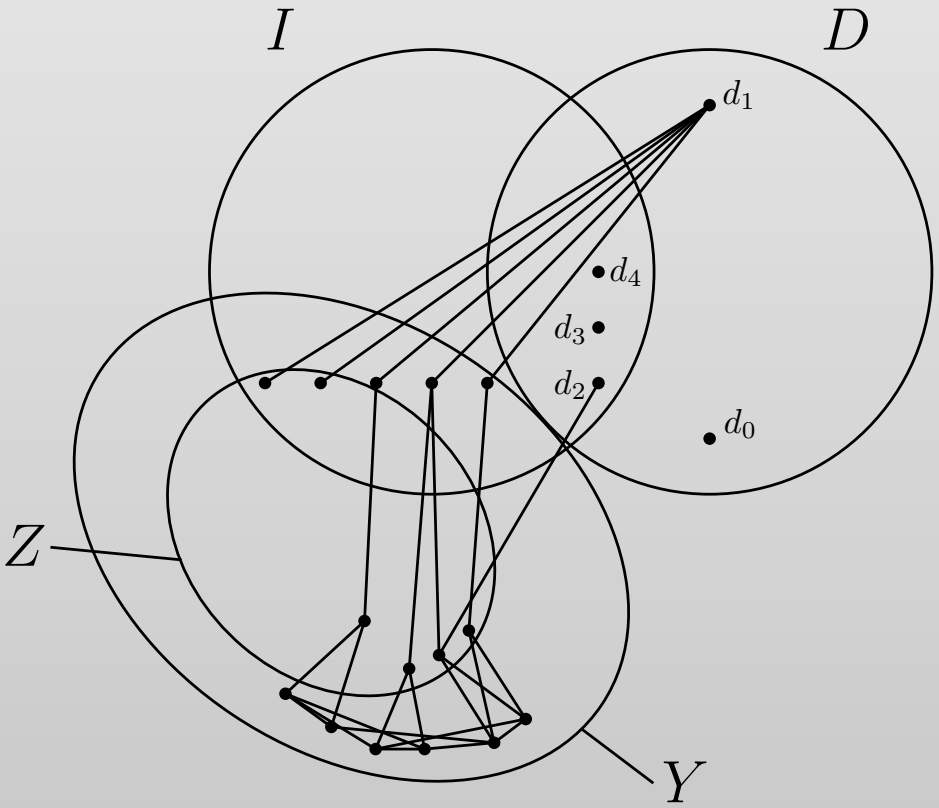

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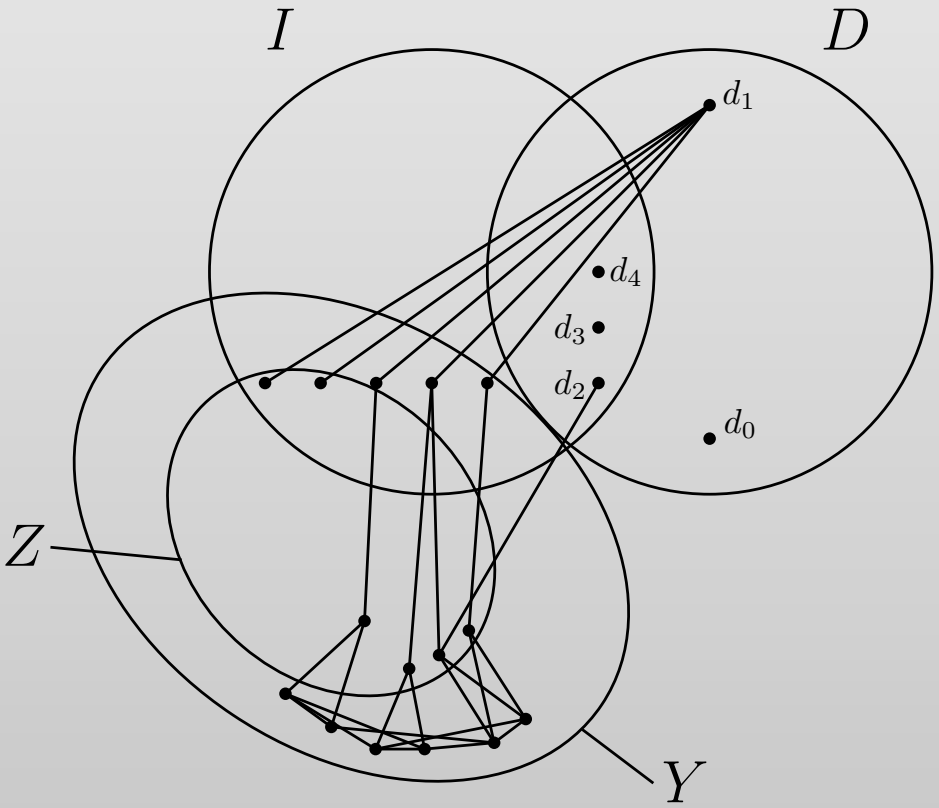

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- Let U be the set of at most $|D \setminus (D' \cup (I \cap D))|$ vertices in $V(G) \setminus D$ adjacent to all of the vertices in $D \setminus (D' \cup (I \cap D))$.
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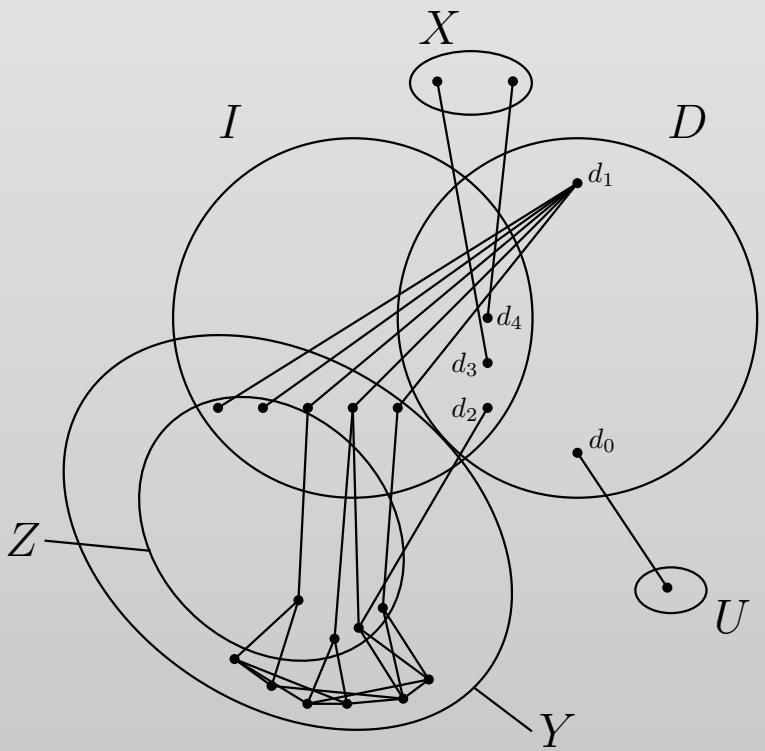

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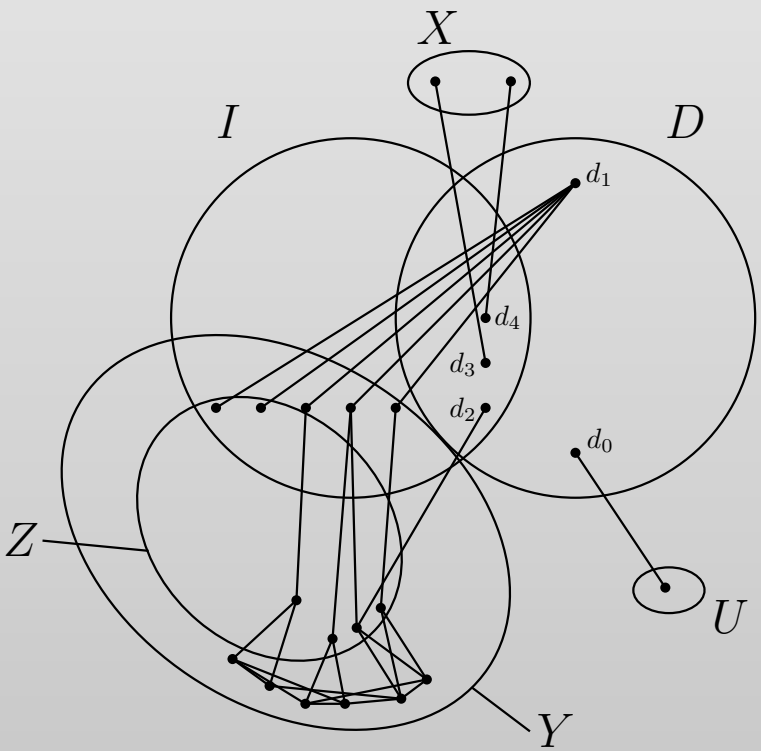

Exemplary Case

- If $|D'| = 0$ or if $|D'| = 2$ then $\Gamma'_D(G) \leq \alpha(G)$ as before.
- If $|Z| \leq |(I \setminus D) \cup D_0| - 1$ then $Z \cup X \cup U$ is an inverse dominating set of an appropriate size.
- If $|Z| > |(I \setminus D) \cup D_0|$ then $Z \cup ((I \cap D) \setminus D_0)$ is a larger independent set than I.
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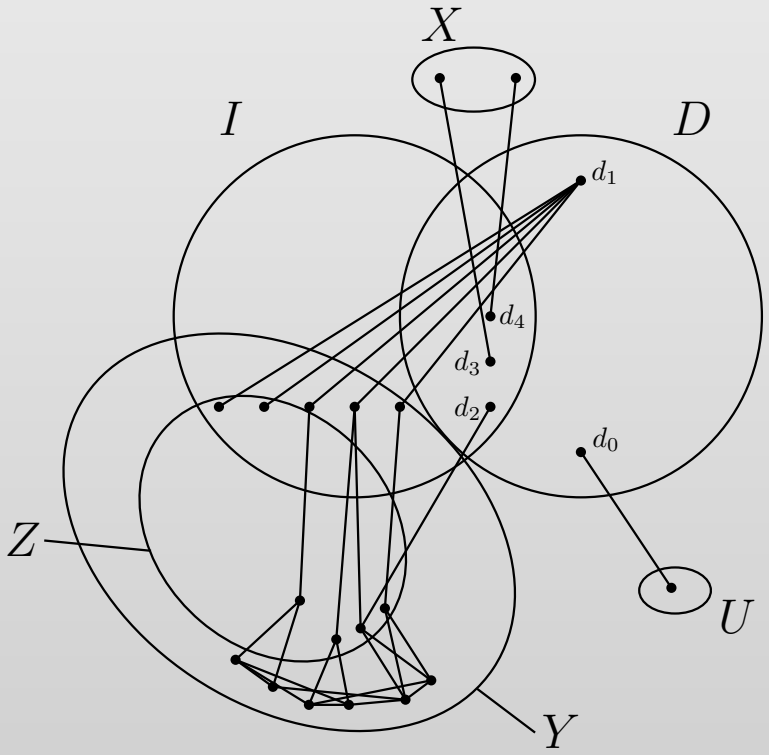

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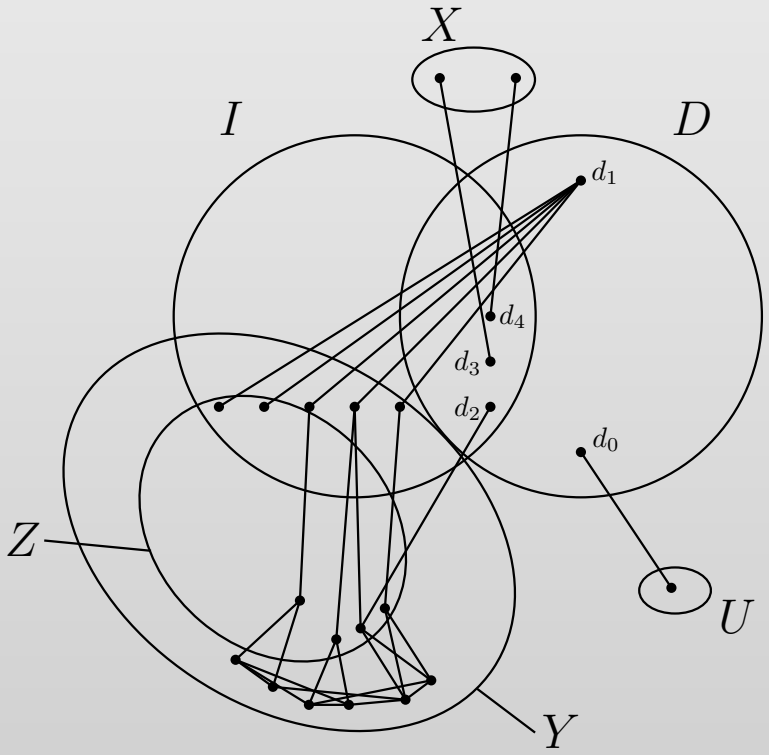

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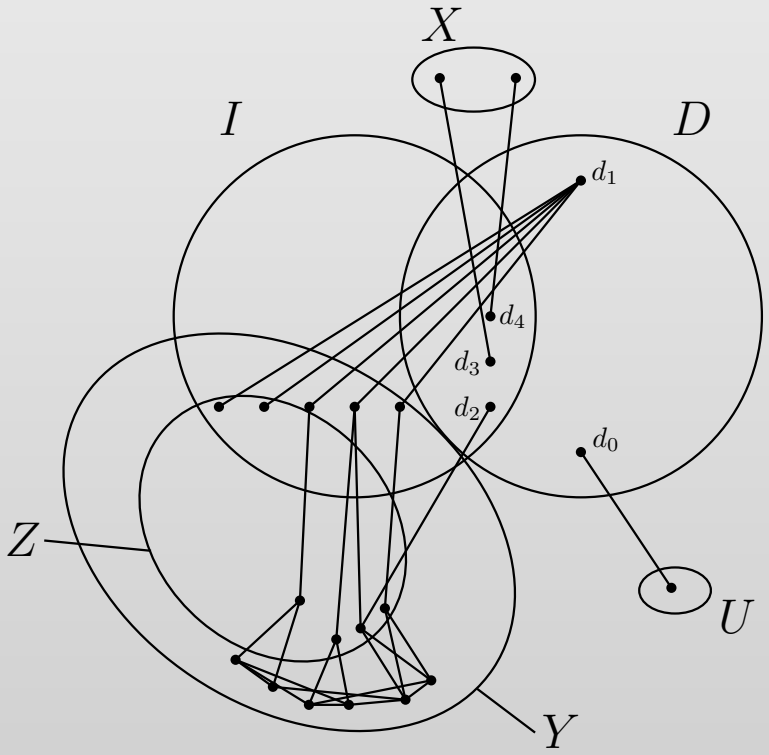

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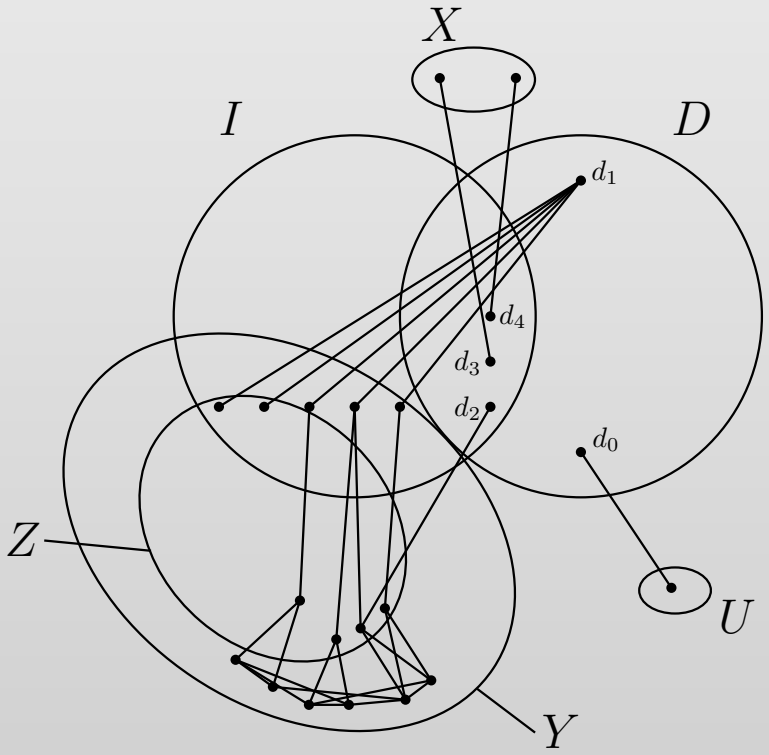

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- The benefit of this approach is that if sub-conjecture is not true, it would point to a possible counterexample.
- This approach is also a plausible approach in that Drs. Peter Johnson and Jessica McDonald (Auburn University) have used this technique to re-prove the inverse domination conjecture for claw-free graphs.

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War Eagle!

Thank You For Your Kind Attention !

Figure: War Eagle !