Möbius Transformations

J. Hammer
Kutztown University of Pennsylvania
james.hammer3@gmail.com

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Abstract

This article delves into properties of Möbius transformations, which are complex functions of the form $T(z) = \frac{az+b}{cz+d}$, for all complex numbers z and for specific complex numbers a, b, c, and d. In order to discuss these intriguing functions with any depth, it is important to begin with a general overview of the complex plane. In that same vein, it is important to give specific definitions as a starting point in the discussion. Similarly, it is important to develop both the geometric and algebraic interpretations of what Möbius transformations do. A few important properties of Möbius transformations will then be described beginning with the claim that every Möbius transformation can be expressed as a composition of translations, inversions, and dilations. In addition, a few important properties of Möbius transformations will be discussed (e.g. the circle preserving property.) The true beauty of Möbius transformations, however, can be seen when one examines the behaviour of the Riemannian Sphere (S^2,) where it can be shown that Möbius transformations can be described as rigid transformations of the sphere. This can be done using two stereographic projections. First, a stereographic projection is used to project the complex plane onto the sphere. Then, a rigid circle preserving transformation is performed on the sphere, followed by a second stereographic projection back onto the complex plane. This argument will be presented by integrating both algebraic and geometric arguments to classify all of the rigid transformations of the unit sphere. Similarly, it shall be shown that Möbius transformations of the extended complex plane correspond to rigid motions of the sphere.

Contents

1 Cardinal Forms

Möbius transformations, also known as linear fractional transformations, are defined as a rational function of the form $\mathfrak{H}(z) = \frac{az+b}{cz+d}$ where a, b, c, and d are complex numbers and $ad - bc \neq 0$ (Fisher, 196). The quantity $ad - bc$ is sometimes called the function's determinant (Schwerdtfeger, 41). The determinant of a Möbius transformation is not allowed to be 0 due to the fact that if it were, the familiar quotient rule for taking derivatives would show that the derivative of $\mathfrak{H}$ would be $\mathfrak{H}'(z) = \frac{ad-bc}{(cz+d)^2} = 0$. This, of course, would mean that the original function, $\mathfrak{H}$, would be a constant function (Fisher, 196). It would be nice if the Möbius transformation was an injective function; for if it were, there would also be an inverse mapping back to the complex plane. This is a fact that will be used later.

Theorem 1.1. *Möbius transformations are injective functions.*

Proof. Let z_1 and z_2 be Complex numbers. Further, assume that $\mathfrak{H}(z_1) = \mathfrak{H}(z_2)$. So,

$$\begin{array}{rccc}
 & \mathfrak{H}(z_1) & = & \mathfrak{H}(z_2) \\
\Longrightarrow & \dfrac{az_1+b}{cz_1+d} & = & \dfrac{az_2+b}{az_2+d} \\
\Longrightarrow & (az_1+b)(az_2+d) & = & (az_2+b)(cz_1+d) \\
\Longrightarrow & acz_1z_2 + adz_1 + bcz_2 + bd & = & acz_1z_2 + adz_2 + bcz_1 + bd \\
\Longrightarrow & adz_1 + bcz_2 & = & acz_1 + adz_2 \\
\Longrightarrow & adz_1 - bcz_1 & = & adz_2 - bcz_2 \\
\Longrightarrow & (ad-bc)\,z_1 & = & (ad-bc)\,z_2 \\
\Longrightarrow & z_1 & = & z_2
\end{array}$$

So, $\mathfrak{H}(z_1) = \mathfrak{H}(z_2)$ implies that $z_1 = z_2$. Moreover, this means that the function $\mathfrak{H}$, a general Möbius transformation, maps distinct points onto distinct images (i.e. it is injective.) (Fisher, 196,197) □

Now that the injectivity of Möbius transformations has been established, it may be helpful to see some basic transformations that can be obtained by Möbius transformations. In this section, four basic transformations will be presented, the translation, the dilation, the rotation, and inversion (which is not technically a Möbius transformation) (Schwerdtfeger, 12). It should be noted at this point that an inversion (geometrically) is not considered a Möbius transformation; rather, it is a special case known as an **anti-homography**, and is of the form $\mathfrak{H}(\overline{z}) = \frac{a\overline{z}+b}{c\overline{z}+d}$ (Schwerdtfeger, 46). In studying these basic transformations, it will be proved that each transformation can be obtained by the composition of two inversions. An inversion can be defined as follows: Let $\mathfrak{C}_0$ be a circle with positive radius and z be a point not on $\mathfrak{C}_0$ nor at its center. Then, there is one and only one point $z^* \neq z$ called the **inverse** of z which is common to all circles through z that are orthogonal to $\mathfrak{C}_0$. The transformation mapping z to z^* is called the **inversion**. $\mathfrak{C}_0$ is called the **fundamental circle** of the inversion. It should be noted that $\mathfrak{C}_0$ can be a straight line, which can be thought of as a circle with infinite radius. In that vain, if $\mathfrak{C}_0$ is a straight line, the point z^* is symmetric to z with respect to $\mathfrak{C}_0$ (Schwerdtfeger, 12).

In addition, it will also be proved that every Möbius transformation is a composition of these four transformations.

Translation. A translation can be visualized as physically moving all the points in the same direction, by the same amount. The translation can be depicted algebraically by the transformations $\mathfrak{T}_b(z) = z+b$, where b is a complex number. Now, it will be shown that the translation is the composition of two inversions.

Theorem 1.2. *The translation $\mathfrak{T}_b(z) = z + b$, is the product of two inversions.*

For proof of this theorem, please see page forty-four of Schwerdtfeger's text.

Rotation. There is nothing mystical about this concept. Rotations fix the origin and spin all other points counter-clockwise by the same amount. Algebraically, rotations are defined as $\mathfrak{R}_\alpha(z) = e^{i\alpha}z$.

Theorem 1.3. *Every rotation of the form $\mathfrak{R}_\alpha(z) = e^{i\alpha}z$ can be expressed as the product of two inversions.*

Further proof of this shall be left to the interested reader in referencing page forty-five of Schwerdtfeger's text.

Dilation. The contraction or magnification of a point in the complex plane is called a dilation. The dilation can be expressed algebraically as $\mathfrak{D}_\rho(z) = \rho z$ for all complex z, where ρ is a positive real number.

Theorem 1.4. *Dilations of the form $\mathfrak{D}_\rho(z) = \rho z$ can be expressed as the product of two inversions.*

Verification of this can be found on pages forty-five and forty-six of Schwerdtfeger's text. Now that the basic transformations have been fleshed out, it can be shown that every Möbius transformation can be expressed as a composition of these functions.

Theorem 1.5. *Every linear fractional transformation is a composition of rotations, translations, dilations, and inversions.*

Proof. Let $\mathfrak{H}(z) = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$. Two cases should be examined. In the first case, assume that $c = 0$. Then, it logically follows that $\mathfrak{H}(z) = \frac{az+b}{d} = \frac{a}{d}z + \frac{b}{d}$, which is clearly a composite of a rotation, a dilation, and a translation. In the second case, assume that $c \neq 0$. Then, it logically follows that

$$\begin{aligned}
\mathfrak{H}(z) &= \frac{az+b}{cz+d} && \mathfrak{H} \\
&= \frac{b+az}{c\left(z+\frac{d}{c}\right)} && \text{Distribution \& Commutativity} \\
&= \frac{bc+acz}{c^2\left(z+\frac{d}{c}\right)} && \text{Multiplication by } c \\
&= \frac{bc-ad+acz+ad}{c^2\left(z+\frac{d}{c}\right)} && \text{Adding } 0 \\
&= \frac{bc-ad+a\left(cz+d\right)}{c^2\left(z+\frac{d}{c}\right)} && \text{Distribution} \\
&= \frac{bc-ad+ac\left(z+\frac{d}{c}\right)}{c^2\left(z+\frac{d}{c}\right)} && \text{Distribution} \\
&= \frac{bc-ad}{c^2} \cdot \frac{1}{z+\frac{d}{c}} + \frac{a}{c} && \text{Distribution.}
\end{aligned}$$

Therefore, every linear fractional transformation is a composition of rotations, translations, dilations, and inversions (Beck, 24). □

2 Circle Preserving Property

Another important characteristic of Möbius transformations is that they are circle preserving. A transformation is called **circle preserving** if it carries straight lines and circles into straight lines and circles (Beck, 21,22).

Theorem 2.1. *Möbius transformations are circle preserving.*

Proof. Since every Möbius transformation is a composition of translations, rotations, dilations, and inversions, each separate case must be considered. Since, translations, rotations, and dilations are obviously circle preserving, it has to be shown that inversions are circle preserving. First, note that the equation of a straight line in the complex plane is $2ax + 2by = c$, where $z = x + yi$, for real number a, b, c, x, and y. Moreover, let $\alpha = a + bi$. Then, it follows that

$\overline{\alpha}z = (a - bi)(x + yi) = ax + by + i(ay - bx)$. Hence,

$$\begin{aligned}
\overline{\alpha}z + \alpha\overline{z} &= [(ax + by) + i(ay - bx)] + [(ax + by) + i(bx - ay)] \\
&= [(ax + by) + i(ay - bx)] + [(ax + by) - i(ay - bx)] \\
&= 2(ax + by) \\
&= 2\mathrm{Re}(\overline{\alpha}z) \\
&= 2c
\end{aligned}$$

More succinctly, the standard form of a line can be expressed as either $\overline{\alpha}z + \alpha\overline{z} = 2c$ or $\mathrm{Re}(\overline{\alpha}z) = c$. Two cases will now be examined. Case 1 will show that circles map into either circles or straight lines; case 2 will show that lines map into either lines or circles.

Case 1: Circles

Given a circle centered at z_0 with a radius r, the equation of the circle is:

$$\begin{array}{lrcll}
 & |z - z_0| &=& r. & \\
\Longrightarrow & |z - z_0|^2 &=& r^2 & \text{Square both sides.} \\
\Longrightarrow & (z - z_0)\overline{(z - z_0)} &=& r^2 & \text{Theorem 1.1.} \\
\Longrightarrow & z\overline{z} - z_0\overline{z} - z\overline{z_0} + z_0\overline{z_0} &=& r^2 & \text{Multiplication.} \\
\Longrightarrow & |z|^2 - z_0\overline{z} - z\overline{z_0} + |z_0| - r^2 &=& 0 & \text{Theorem 1.1.}
\end{array}$$

Now, the inverse transformation $w = \dfrac{1}{z}$ yields $z = \dfrac{1}{w}$, and $\overline{z} = \dfrac{1}{\overline{w}}$. Substitution yields

$$\left|\frac{1}{w}\right|^2 - z_0\frac{1}{\overline{w}} - \overline{z_0}\frac{1}{w} + |z_0|^2 - r^2 = 0$$

Multiplying both sides of the equation by $|w|^2 = w\overline{w}$ shows that the preceding equation is equivalent to the following equation:

$$1 - z_0 w - \overline{z_0 w} + |w|^2\left(|z_0|^2 - r^2\right) = 0$$

Two sub-cases now have to be examined.

Sub-case 1: If $r = |z_0|$, then the equation becomes $z_0 w + \overline{z_0 w} = 1$. Letting $\alpha = \overline{z_0}$, the previous equation becomes $\overline{\alpha}w + \alpha\overline{w} = 1$, which is clearly the equation of a line.

Sub-case 2: If $r \neq |z_0|$, then dividing through by $|z_0|^2 - r^2$ produces

$$|w|^2 - \frac{z_0}{|z_0| - r^2}w - \frac{\overline{z_0}}{|z_0| - r^2}\overline{w} + \frac{1}{|z_0| - r^2} = 0$$

Letting $w_0 = \dfrac{\overline{z_0}}{|z_0| - r^2}$ and $s^2 = |w|^2 - \dfrac{1}{|z_0| - r^2} = \dfrac{|z_0|^2}{(|z_0| - r^2)} - \dfrac{|z_0| - r^2}{(|z_0| - r^2)} = \dfrac{r^2}{(|z_0| - r^2)}$ allows the previous equation to be written as

$$\begin{array}{rrcll} & |w|^2 - \overline{w_0}w - w_0\overline{w} + |w|^2 - s^2 & = & 0 & \text{Substitution} \\ \Longrightarrow & w\overline{w} - w_0\overline{w} - w\overline{w_0} + w_0\overline{w_0} & = & s^2 & \text{Theorem 1.1.} \\ \Longrightarrow & (w - w_0)\,\overline{(w - w_0)} & = & s^2 & \text{Theorem 1.1.} \\ \Longrightarrow & |w - w_0|^2 & = & s^2 & \text{Theorem 1.1.} \end{array}$$

Which is the equation of a circle centered at w_0 with a radius of s.

Case 2: Lines

Starting with the equation of a line, substitute the transformation $w = \dfrac{1}{z}$ to produce the following:

$$\begin{array}{rrcll} & \overline{z_0}\dfrac{1}{w} + \dfrac{1}{\overline{w}}z_0 & = & 2c & \text{Substitution} \\ \Longrightarrow & \overline{z_0}\overline{w} + z_0 w & = & 2cw\overline{w} & \text{Multiplication by } |w|^2. \end{array}$$

Once again, two sub-cases must be evaluated.

Sub-case 1: If $c = 0$, then $\overline{z_0}\overline{w} + z_0 w = 2cw\overline{w}$ is $\overline{z_0}\overline{w} + z_0 w = 0$, which is the equation of a line in terms of w.

Sub-case 2: If $c \neq 0$, then dividing by $2c$ shows that

$$\begin{array}{rrcll} & w\overline{w} - \dfrac{\overline{z_0}}{2c}\overline{w} - \dfrac{z_0}{2c}w & = & 0 & \text{Division by } 2c \\ \Longrightarrow & \left(w - \dfrac{\overline{z_0}}{2c}\right)\left(\overline{w} - \dfrac{z_0}{2c}\right) - \dfrac{|z_0|^2}{4c^2} & = & 0 & \text{Completing the Square.} \\ \Longrightarrow & \left|w - \dfrac{\overline{z_0}}{2c}\right|^2 & = & \dfrac{|z_0|^2}{4c^2} & \text{Theorem 1.1.} \end{array}$$

Which is the equation of a circle centered at $\frac{\overline{z_0}}{2c}$ with the radius of $\frac{|z_0|}{2|c|}$.

Henceforth, Möbius transformations are circle preserving (Beck, 24, 25). □

3 Relationship to the Sphere

Possibly the most interesting thing about Möbius transformations is how they are related to the sphere. In general, Möbius transformations can be produced by projecting an image onto the sphere via a stereographic projection, followed by a rigid rotation of the sphere, and concluded by another stereographic projection back down to the complex plane. It is the goal of this section to prove that fact; however, some prerequisites are needed. More specifically, the equations of the stereographic projection will be derived. This will be followed by a discussion on how the stereographic projection treats circles and lines both in the plane and on the sphere. The remaining proofs will flesh out the notion that a rigid motion of the sphere corresponds to a Möbius transformation on the complex plane.

The stereographic projection is quite possibly the most intuitive projection from the extended complex plane (the complex plane with the concept of a **point at infinity** added on) into the sphere. The extended complex plane shall be denoted by the symbol $\hat{\mathbb{C}}$. Geometrically speaking, the stereographic projection can be achieved by first allowing the extended complex plane to cut through the equator of the unit sphere, S^2. Then, in order to map every point onto the sphere, one must draw a straight line from the point on the complex plane to the north pole, N, of the sphere. This is easy to see for the points outside of the sphere; however, it is not so intuitive as to what happens to points inside of the sphere. For points inside of the sphere, one must simply do as before in drawing a straight line from the complex plane onto N, only this time, extending the line down to the southern hemisphere. Just as before, the intersection of the line drawn from z in the complex plane to N with the sphere is the image of z. The point at infinity will be used in order to cover the north pole of the sphere. Intuitively, it is clear that the stereographic projection is continuous. Since the complex plane is continuous, and the stereographic projection simply maps every point in the complex plane to the sphere, the projection should also preserve continuity. The only point of concern would be the north pole; however, since that is being covered by the point at infinity, the projection should remain continuous. Similarly, it is easy to see that as points in the complex plane get closer and closer to infinity, the image on the sphere gets closer and closer to the north pole. A similar argument can be made from the sphere to the complex plane.

It will prove beneficial to determine the equation of the stereographic projection at this point. The grueling details of this will be left to appendix A1; however, the moral of the exercises is that when one is going from the plane to the sphere, the equation looks like $z = \frac{\xi + i\eta}{1+\zeta}$. Similarly, a point in space, $P = (\xi, \eta, \zeta)$ has the coordinates $\xi + i\eta = \frac{2z}{1+z\bar{z}}$ and $\zeta = \frac{1-z\bar{z}}{1+z\bar{z}}$. Now that the equations for the stereographic projection have been taken care of, it shall be shown that the stereographic projection respects circles and lines from the extended complex plane to the sphere and vice versa.

Theorem 3.1. *The stereographic projection carries circles and lines of the plane into circles and "lines" on the sphere and conversely (Schwerdtfeger, 24).*

Schwerdtfeger presents an eloquent proof of this on page twenty-four of his text.

Now that it has been established that the stereographic projection carries circles and lines from the plane to the sphere as well as from the sphere to the plane, the question arises as to what kind of transformations of the sphere respect circles and lines in the same way. The easiest circle preserving transformations of the sphere are the rigid motions of the sphere. A **rigid motion** is a mapping where any pair of image points have the same distance as the corresponding pair of inverse image points (Kreyszig, 4). Clearly rotating the sphere in any direction will preserve the distance between any two points. Similarly, translations of the sphere will preserve the distance between any two points. Lastly, a reflection of the unit sphere will also preserve the distance between any two points in a similar way. It is the goal of this remaining paper to drive home the idea that a Möbius transformation can be described in terms of a rigid transformation of the sphere followed by a stereographic projection down to the complex plane.

Theorem 3.2. *Let there be given three different points, z_1, z_2, and z_3 in the complex plane, then the three points, Z_1, Z_2, and Z_3, where Z_i represents the image of z_i, i.e. $Z_1 = \mathfrak{H}(z_1), Z_2 = \mathfrak{H}(z_2)$, and $Z_3 = \mathfrak{H}(z_3)$, are all that is needed to ensure that the Möbius transformation, $\mathfrak{H}$, is uniquely determined (Schwerdtfeger, 47).*

A thorough proof of this can be found on page forty seven of Schwerdtfeger's text.

Theorem 3.3. *Every injective function mapping the complex plane into itself that is circle preserving is either a Möbius transformation or an anti-homography (Schwerdtfeger, 106).*

A more elaborate version of Schwerdtfeger's proof of this follows:

Proof. Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be an injective function. Define a Möbius transformation, $\mathfrak{H}$, that maps $f(0), f(1)$, and $f(\infty)$ onto $0, 1$, and ∞ respectively. Moreover, define $g(z)$ as either $\mathfrak{H} \circ f(z)$ or $\overline{\mathfrak{H} \circ f(z)}$, making sure that the imaginary part of $g(i)$ is greater than or equal to 0. The goal is to show that $g(z) = z$ for all z in the extended complex plane.

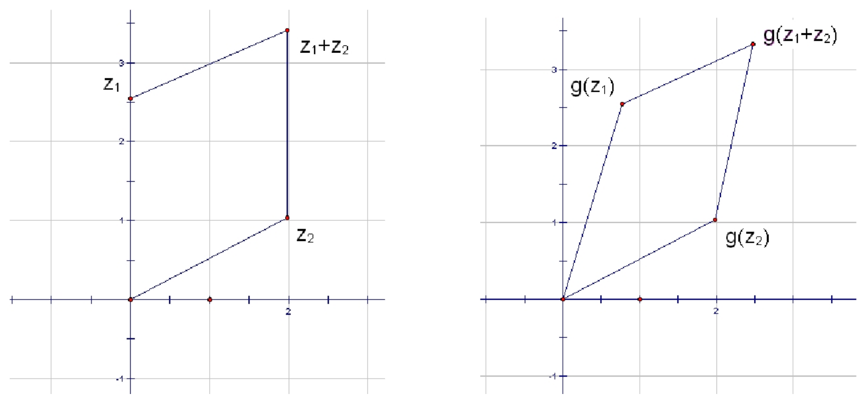
First, one must prove that g maps the real axis onto itself. Because g is circle preserving and maps the point at infinity to the point at infinity, g also maps lines to lines. Since the fixed points of g are 0 and 1, the real axis gets mapped to itself.

Second, it must be shown that g maps parallel lines to parallel lines. Assume that l and m are two parallel lines in the extended complex plane. Then both l and m contain the point at infinity. Since g maps the point at infinity to itself, $g(l)$ and $g(m)$ contain the point at infinity. Hence, $g(l)$ is parallel to $g(m)$.

Third, it must be shown that g maps orthogonal lines to orthogonal lines. Consider two perpendicular lines, l and m. Let l^* and m^* be lines such that l is parallel to l^*, $l \neq l^*$ and m parallel to $m^*, m \neq m^*$. Then, l, m, l^*, and m^* form a rectangle with vertices z_1, z_2, z_3, and z_4. By the second concept, $g(l)$ is parallel to $g(l^*)$ and $g(m)$ is parallel to $g(m^*)$. Hence, $g(z_1), g(z_2), g(z_3)$ and $g(z_4)$ are the vertices of a parallelogram. Moreover, there exists a circle C through z_1, z_2, z_3, and z_4; so, $g(C)$ is a circle containing $g(z_1), g(z_2) g(z_3)$, and

$g(z_4)$. Consequently, $g(z_1), g(z_2)g(z_3)$, and $g(z_4)$ are vertices of a rectangle, i.e. $g(l)$ is perpendicular to $g(m)$.

Fourth, g is additive, i.e. $g(z_1) + g(z_2) = g(z_1 + z_2)$, for all complex numbers z_1 and z_2. If $0, z_1$, and z_2 are not collinear, then $0, z_1, z_2$, and $z_2 + z_1$ get mapped to the parallelogram with vertices $0, g(z_1), g(z_2)$, and $g(z_2 + z_1)$ respectively. i.e.



It is clear that

$$g(z_1) + g(z_2) = g(z_1 + z_2).$$

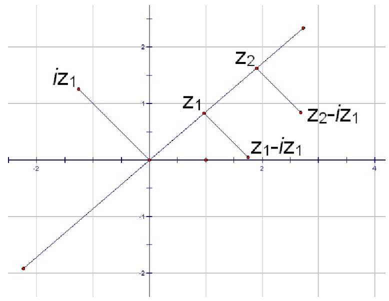
Similarly,

$$g(z_1) - g(z_2) = g(z_1 - z_2).$$

If z is a complex number that is not 0, then there exists a complex number z_1 such that $0, z$, and z_1 are non-collinear. Then it immediately follows that

$$\begin{array}{lrcll} & g(z_1) - g(z) & = & g(z_1 - z) & \text{Above} \\ & & = & g(z_1 + (-z)) & \text{Additive Inverse} \\ & & = & g(z_1) + g(-z) & \text{Part Four} \\ \text{Hence,} & g(-z) & = & -g(z) & \text{For all } z. \end{array}$$

Now, if $0, z_1$, and z_2 are collinear, there are two sub-cases. If $z_1 = -z_2$, then the above case shows that $g(z_1) = -g(z_2)$. Hence, $g(z_1 + z_2) = g(0)$ and $g(z_1) + g(z_2) = 0$. For the second and final sub-case: if $z_1 \neq -z_2$, then $z_1 - iz_1$ and $z_2 + iz_1$ do not lie on the line with 0.



Then, the previous case can be applied as follows:

$$\begin{aligned}
g(z_1+z_2) &= g\left((z_1-iz_1)+(z_2+iz_1)\right) && \text{Substitution}\\
&= g\left(z_1-iz_1\right)+g\left(z_2+iz_1\right) && \text{Argument Four}\\
&= g(z_1)-g(iz_1)+g(z_2)+g(iz_1) && \text{Argument Four}\\
&= g(z_1)+g(z_2) && \text{Addition}
\end{aligned}$$

Fifth, it follows from the fourth argument by a straight forward induction that for all integers, n, and for all complex numbers z,

$$g(nz)=ng(z).$$

Step six, show that for all rational numbers r and for all complex z, $g(rz)=rg(z)$. Let r be a rational number, i.e. $r=\frac{n}{m}$, for integers m and n. Then

$$g(z)=g\left(m\frac{z}{m}\right)=mg\left(\frac{z}{m}\right).$$

Hence,

$$\begin{aligned}
g(rz) &= g\left(\frac{n}{m}z\right) && \text{Substitution}\\
&= g\left(n\frac{z}{m}\right) && \text{Algebra}\\
&= ng\left(\frac{z}{m}\right) && \text{Step five}\\
&= \frac{n}{m}g(z) && \text{Step five}\\
&= rg(z) && \text{Substitution}
\end{aligned}$$

It is clear to see from the above argument that $g(r)=r$, for all rational numbers, r.

Step seven, Since the real axis is perpendicular to the imaginary axis, the images of the two axes are also perpendicular by step two. Since $g(\mathbb{R})=\mathbb{R}$, $g(i\mathbb{R})=i\mathbb{R}$, i.e. g maps the imaginary axis to itself. Hence, for some real number λ,

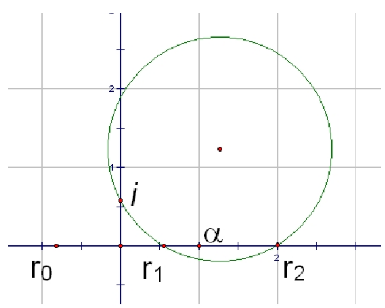
$$g(i)=\lambda i.$$

Now, by step four, it follows that $g(1+i) = g(1)+g(i) = 1+\lambda i$, and $g(-1+i) = -1+\lambda i$. Since l is perpendicular to m, it follows that the images of l and m are perpendicular by step two. The angle created by the points $1+\lambda i, 0$, and $-1+\lambda i$ has to be ninety degrees. Hence, $0 = (1,\lambda)\cdot(-1,\lambda) = -1+\lambda^2$, i.e. $\lambda^2 = 1$ or $\lambda = \pm 1$. Since g was chosen so that $\mathrm{Im}(g(i)) \geq 0$, it follows that $\lambda = 1$, i.e. $g(i) = i$.

Step eight: Show that for all rational numbers r and s, $g(r+si) = r+si$

$$\begin{aligned} g(r+si) &= g(r)+g(si) \\ &= r+sg(i) \\ &= r+si. \end{aligned}$$

Step nine: Show that for all irrational α, $g(\alpha) = \alpha$. Let α be an irrational number, and let r_0, r_1, and r_2 be rational such that $r_0 < r_1 < \alpha < r_2$. Since r_1, r_2, and i are fixed points of g, and three points uniquely define a circle, it follows that $g(C) = C$.



Note that r_0 is outside of C and α is inside. Hence, any circle C^* through r_0 and α will intersect C, which implies that $g(C^*)$ also intersects $C = g(C)$. Hence $g(\alpha)$ lies in the interior of C and since $g(\alpha)$ is a real number, it follows that $r_1 < g(\alpha) < r_2$. This is true for any choice of r_1 and r_2, hence it follows that $g(\alpha) = \alpha$. Note that this can be seen by a contradiction, where one would assume that $g(\alpha) \neq \alpha$ and find r_1 and r_2 such that $r_1 < \alpha < r_2 < g(\alpha)$ (or $g(\alpha) < r_1 < \alpha < r_2$.) This would contradict our assumption.

Step ten: Show that for all irrational α, $g(\alpha i) = \alpha i$. This would be done with the same argument rotated by ninety degrees.

Step eleven: Everything up to this point culminates to

$$\begin{aligned} g(z) &= g(x+iy) \\ &= g(x)+g(iy) \\ &= x+iy \\ &= z \end{aligned}$$

Hence $g(z) = z$ for all z in the extended complex plane.

Henceforth, it follows that for all z in the extended complex plane, either

$$\mathfrak{H} \circ f(z) = z,$$

which means that

$$f(z) = \mathfrak{H}^{-1}(z),$$

which is a Möbius transformation, or

$$\overline{\mathfrak{H} \circ f(z)} = z,$$

which means that

$$f(z) = \mathfrak{H}^{-1}(\overline{z}),$$

which is the equation for an anti-homography. Thus, every injective function mapping the complex plane into itself that is circle preserving is either a Möbius transformation or an anti-homography □

In plain English, theorem 3.3 says that the only circle preserving transformations in the extended complex plane are Möbius transformations and anti-homographies. Three very interesting corollaries follow directly from this proof. Together, they will show that the rigid motions of the unit sphere that preserve orientation at a point correspond to Möbius transformations in the extended complex plane. Similarly, the transformations that do not preserve orientation will correspond to anti-homographies in the extended complex plane.

Corollary 3.4. *Every circle preserving transformation of the completed plane, which preserves the sense of rotation (orientation) at one point, is necessarily a Möbius transformation (Schwerdtfeger, 109).*

Proof. Theorem 3.3 has just proved that the only circle preserving transformations of the complex plane onto itself are Möbius transformations or anti-homographies. The anti-homographies, however, are comprised of either a reflection followed by a Möbius transformation or a Möbius transformation followed by a reflection. In either case, the reflection changes the orientation of the images; therefore, every circle preserving transformation of the completed plane, which preserves the sense of orientation at one point, is necessarily a Möbius transformation. □

Corollary 3.5. *A rigid motion of the sphere that does not change orientation corresponds to a Möbius transformation in the extended complex plane.*

Proof. Rigid transformations of the sphere that do not change orientation will remain circle preserving transformations that also preserve orientation through the stereographic projection. Therefore, the transformation in the extended complex plane will be a circle preserving transformation that respects orientations. Moreover, by the above corollary, a rigid transformation of the sphere that does not change orientation corresponds to a Möbius transformation in the extended complex plane. □

Corollary 3.6. *A rigid transformation of the sphere that changes orientation corresponds to an anti-homography (i.e. a reflection followed by a Möbius transformation) in the extended complex plane.*

Proof. As in the above corollary, a rigid transformation of the sphere that changes orientation will correspond to a circle preserving transformation of the extended complex plane that changes orientation. Since anti-homographies are either reflections followed by a Möbius transformation or vice versa, they change the orientation from the pre-image to the image. By theorem 3.3, since the only two circle preserving transformations of the extended complex plane are Möbius transformations and anti-homographies, the rigid transformations of the sphere that change orientation correspond to an anti-homographies in the extended complex plane. □

More intuitively, it means that every Möbius transformation or anti-homography can be expressed by first projecting the pre-image of the extended complex plane onto the sphere via a stereographic projection. This must then be followed by a rigid transformation of the sphere, and concluded by a stereographic projection back onto the extended complex plane. Those rigid transformations of the sphere which are orientation preserving (i.e. those rigid transformations that are not reflections) correspond to Möbius transformations on the extended complex plane. Similarly, the rigid transformations of the sphere which do not preserve orientation (i.e. reflections) correspond to anti-homographies in the extended complex plane. At this point, one can plainly see the elegance of Möbius transformations and anti-homographies.

4 Sphere to Plane

Above, it has been shown that rigid motions of the unit sphere in three space correspond to Möbius transformations and anti-homographies in the complex plane via a stereographic projection; however, the other direction has yet to be shown. Namely, that Möbius transformation in the extended complex plane correspond to rigid transformations of the unit sphere in three space. To do this, however, the definition of a stereographic projection must be slightly altered.

To rectify the stereographic projection, one must allow the extended complex plane to first lay tangent to the unit sphere at the south pole (S.) Then, one must allow transformations which may move the unit sphere some height, h, above the extended complex plane. It is important to realize that the only thing that changes with the stereographic projection is the area of the complex plane which is being mapped to the south pole. Recall, when the complex plane cut the unit sphere at it's equator, only the points that were within the sphere were mapped to the south pole of the sphere. Now that the complex plane lies tangent to the south pole, the radius of points in the complex plane which maps to the southern hemisphere is greater. Similarly, when the sphere is raised a certain height above the complex plane, the radius of points which are mapped to the south pole continues to increase. Moreover, everything accept the equations of the stereographic projection should stay the same. The new equations, however, will prove helpful in the following proofs. Manipulating the equations for the stereographic projection as in appendix A-1 shows that if one were to lower the complex plane so that it lies tangent to the south pole of the unit sphere, one would see that $\zeta + i\eta = \frac{4}{4+z\overline{z}}(x+iy)$ and $\xi = \frac{z\overline{z}-4}{4+z\overline{z}}$, where (ζ, η, ξ) is a point in three space and z is a point in the complex plane. Similarly, using the same conventions, one will find that $z = \frac{-2(\zeta+i\eta)}{\xi-1}$.

5 Conclusion

In Summary, Möbius transformations are incredibly fascinating and beautiful functions. Their true beauty resides in the stereographic projection to the unit sphere. In studying the relationship between Möbius transformations in the complex plane and their relationship to the unit sphere, one begins to appreciate the interconnectedness of many fields of mathematics, including complex analysis, geometry, and algebra.

Möbius transformations are functions that map the complex plane into itself, which are composed of four basic types of transformations. They are composed of translations, rotations, dilations, and inversions. The inversions, however, are not considered Möbius transformations because they do not preserve orientations. Rather, they are called anti-homographies, and can be viewed either as a reflection followed by a Möbius transformation or vice versa. Nevertheless, it has also been proved that Möbius transformations are all circle preserving, meaning pre-image circles and lines are mapped into circles and lines. Once this has been established, it is important to examine the stereographic projection from the extended complex plane to the unit sphere. Once again, the circle preserving property must be proved with the equations for the stereographic projection. To show that the rigid transformations of the sphere correspond to Möbius transformations and anti-homographies on the extended compex plane, one must first show that there are no other circle preserving transformations in the complex plane. Once this is established, it can easily be shown through several corollaries that the rigid transformations of the sphere which preserve orientation correspond to Möbius transformations in the extended complex plane. Similarly, rigid transformations of the sphere which do not preserve orientation correspond to anti-homographies in the extended complex plane. Similarly, it has been shown that Möbius transformations correspond to rigid transformations of the sphere in three space, meaning that the rigid transformations of the unit sphere in three space are isomorphic to Möbius transformations in the extended Complex plane.

There are many more facts to be proved regarding Möbius transformations and anti-homographies that are not covered in this article. For example, the fact that the set of all Möbius transformations forms a group has been omitted from this article in order to preserve the focus of this paper. Conformality, which is an angle preserving property between curves, is another interesting fact about Möbius transformations which was not in the scope of this project (Fisher, 208). In short, there is a lot of research to be done; however, a few quite elegant properties have been fleshed out throughout this article.

APPENDIX

A-1 Equation of the Stereographic Projection

It will prove beneficial to determine the equation of the stereographic projection at this point. To that end, let (ξ, η, ζ) denote a Cartesian coordinate system. Then, it follows that $\xi^2 + \eta^2 + \zeta^2 = 1$ represents the unit sphere. Moreover, let the extended complex plane go through the $\xi\eta$ plane such that for a point $z = x + iy, \xi = x, \eta = y$, and $\zeta = 0$. Two points in the space will be needed, call them $P_1 = (\xi_1, \eta_1, \zeta_1)$ and $P_2 = (\xi_2, \eta_2, \zeta_2)$. Then, any point $Q = (\xi, \eta, \zeta)$ on the line $\overleftrightarrow{P_1P_2}$ is of the form $Q = (1-\lambda)\,P_1 + \lambda P_2$, where λ is a real number. More specifically,

$$Q = \begin{cases} \xi &= (1-\lambda)\,\xi_1 + \lambda\xi_2 \\ \eta &= (1-\lambda)\,\eta_1 + \lambda\eta_2 \\ \zeta &= (1-\lambda)\,\zeta_1 + \lambda\zeta_2 \end{cases}$$

Now, to find a generic point, P of $z = x + iy$, let $P_1 = N = (0, 0, 1)$, and $P_2 = (x, y, 0)$. Then, $P = Q = (\xi, \eta, \zeta) = (\lambda x, \lambda y, (1-\lambda))$. Because this is a point on the unit sphere, it must satisfy the equation $\xi^2+\eta^2+\zeta^2 = 1$. Moreover,

$$\begin{aligned} \xi^2 + \eta^2 + \zeta^2 &= \lambda^2x^2 + \lambda^2y^2 + 1 - 2\lambda + \lambda^2 && \text{Substitution} \\ &= \lambda^2\left(x^2 + y^2 + 1\right) - 2\lambda + 1 && \text{Associativity \& Distribution} \\ &= 1 && \text{Criteria.} \end{aligned}$$

Subtracting 1 from both sides shows that $0 = \lambda^2\left(x^2 + y^2 + 1\right) - 2\lambda$. Solving for lambda,

$$\begin{aligned} \lambda &= \frac{2 \pm \sqrt{4 - 4\,(x^2 + y^2 + 1)\,(0)}}{2\,(x^2 + y^2 + 1)} && \text{Quadratic Formula} \\ &= \frac{2 \pm \sqrt{4}}{2\,(x^2 + y^2 + 1)} && \text{Multiplication by 0} \\ &= 0, \frac{2}{x^2 + y^2 + 1} && \text{Definition of Square Root \& } \pm \end{aligned}$$

The occasion where $\lambda = 0$, however, corresponds to the north pole. So, $P = (\xi, \eta, \zeta)$ is given by

$$\begin{aligned}
\xi + i\eta &= \lambda x + i\lambda\eta && \text{Substitution} \\
&= \frac{2}{x^2+y^2+1}x + i\frac{2}{x^2+y^2+1}y && \text{Substitution} \\
&= \frac{2}{x^2+y^2+1}(x+iy) && \text{Distribution} \\
&= \frac{2z}{z\overline{z}+1} && \text{Theorem 1.1 \& Definition of } z
\end{aligned}$$

Similarly,

$$\begin{aligned}
\zeta &= (1-\lambda) && \text{Definition} \\
&= \left(1 - \frac{2}{x^2+y^2+1}\right) && \text{Substitution} \\
&= \frac{x^2+y^2+1-2}{x^2+y^2+1} && \text{Multiplication by 1} \\
&= \frac{x^2+y^2-1}{x^2+y^2+1} && \text{Simpification} \\
&= \frac{z\overline{z}-1}{z\overline{z}+1} && \text{Theorem 1.1}
\end{aligned}$$

Finally, to find the point z in the complex plane for P, let $P_1 = N$, $P_2 = P$, and $Q = (x, y, 0)$. Then,

$$x = \lambda\xi, \quad y = \lambda\eta, \quad 0 = (1-\lambda) + \lambda\zeta.$$

This means that $\zeta = \dfrac{\lambda - 1}{\lambda}$. Solving for λ in terms of ζ,

$$\begin{aligned}
\lambda &= \frac{1}{\frac{1}{\lambda}} && \text{Reciprocal} \\
&= \frac{1}{\frac{\lambda+1-\lambda}{\lambda}} && \text{Addition by 0} \\
&= \frac{1}{1+\frac{1-\lambda}{\lambda}} && \text{Simplification} \\
&= \frac{1}{1-\frac{\lambda-1}{\lambda}} && \text{Additive Inverse} \\
&= \frac{1}{1-\zeta} && \text{Substitution.}
\end{aligned}$$

Henceforth,

$$\begin{aligned}
z &= x + iy && \text{Definition} \\
&= \lambda\xi + i\lambda\eta && \text{Substitution} \\
&= \frac{1}{1-\zeta}\xi + i\frac{1}{1-\zeta}\eta && \text{Substitution} \\
&= \frac{\xi + i\eta}{1-\zeta} && \text{Common Denominator.}
\end{aligned}$$

Now that the equations for the stereographic projection have been fleshed out, it shall be shown that the stereographic projection respects circles and lines from the extended complex plane to the sphere and vice versa.

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