1 Maximum Degrees of Iterated Line Graphs

Note. All graphs in this section are simple.

Problem 1. *A simple graph G is promising if and only if G is not terminal.*

1.1 Lemmas

Notation. We denote the line graph of G by $L(G)$ and the n-th line graph by $L^n(G)$.

Lemma 1. *If $\Delta(G) \geq 1$, then $\Delta(G) - 1 \leq \Delta(L(G)) \leq 2\Delta(G) - 2$.*

Lemma 2. *If $\Delta(G) \geq 2$, then $\Delta(L^2(G)) \geq 2\Delta(G) - 4$.*

1.2 Definitions

Definition 1. *A simple graph G is said to be **terminal** if $L^k(G)$ is not defined for some $k \in \mathbb{N}$.*

Definition 2. *A simple graph G is said to be **limited** if $\Lambda(G)$ is finite, where $\Lambda(G) = \{L^k(G) | k \in \mathbb{Z}^+\}$.*

Definition 3. *A simple graph G is said to be **nice** if $\Delta(L(G)) = 2\Delta(G) - 2 \geq 0$.*

Definition 4. *A simple graph G is said to be **swell** if every element of $\Lambda(G)$ is nice.*

Definition 5. *A simple graph G is said to be **promising** if $\Lambda(G)$ contains a swell graph.*

1.3 Theorems

Theorem 1. *G is terminal if and only if every component of G is a path.*

Theorem 2. *G is limited if and only if every component of G is a path, a cycle, or $K_{1,3}$.*

Theorem 3. *G is nice if and only if $\Delta(C(G)) \geq 1$ if and only if $C(L(G)) = L(C(G))$.*

Theorem 4. *G is swell if and only if $C(G)$ is not terminal.*

Conjecture 1. *G is promising if and only if G is not terminal.*

2 Moore Graphs

Problem 2. *Does a Moore graph with girth 5 and degree 57 exist?*

2.1 Definitions

Definition 6. *A **Moore graph**, G, is a regular graph of degree d and diameter (i.e. the longest shortest path in a graph) k such that*

$$|V(G)| = 1 + d \sum_{i=0}^{k-1} (d-1)^i$$

.

Remark 1. *An equivalent definition of a Moore graph is that it is a graph of diameter k with girth $2k + 1$.*

Remark 2. *As well as having the maximum possible number of vertices for a given combination of degree and diameter, Moore graphs have the minimum possible number of vertices for a regular graph with given degree and girth. (i.e. Moore graphs form a cage.)*

2.2 Examples

- Any odd cycle is a Moore graph with degree 2.
- The Petersen graph is a Moore graph with degree 3, diameter 2, and girth 5.
- The Hoffman-Singleton graph is a Moore graph with degree 7, diameter 2, and girth 5.

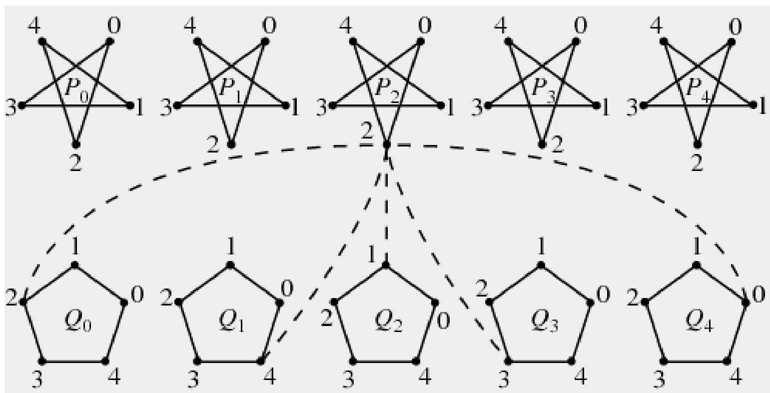

Figure 1: Hoffman-Singleton Graph

Conjecture 2. *There is a Moore graph with girth* 5 *and degree* 57. *A construction of this graph would be awesome.*

3 WarEagle-Regular graphs

Problem 3. *Can anything be said about the form of WarEagle-Regular graphs?*

3.1 Definitions

Definition 7. *A **Strongly-Regular** graph, $SR(n, d, \lambda, \mu)$, is a graph on n vertices that is regular of degree d with the property that every two adjacent vertices have exactly λ common neighbors, and every two non-adjacent vertices have exactly μ common neighbors.*

Definition 8. *An **Edge-Regular** graph, $ER(n, d, \lambda)$ is a graph on n vertices that is regular of degree d with the property that every two adjacent vertices have exactly λ common neighbors.*

Remark 3. *Unlike Strongly regular graphs, we are not concerned with non-adjacent vertices in the Edge-Regular Graphs. Because of this, it is clear that the class of Edge-Regular graphs is a superset of the class of Strongly Regular graphs; so, we cannot dream of classifying these graphs. That doesn't mean that we cannot find some interesting properties that these graphs might share.*

Definition 9. *A **WarEagle-Regular** graph, $WE(n, d, \mu)$ is a graph on n vertices that is regular of degree d with the property that every two non-adjacent vertices have exactly μ common neighbors.*

Remark 4. *Similar to the Edge-Regular graphs, the WarEagle-Regular graphs are a superset of the Strongly regular graphs. We do not wish to classify these graphs; however, it would be nice to get a feel for what they have to look like. Perhaps we can find some neat properties that are forced onto these graphs.*

4 Total Coloring

Problem 4. *The total chromatic number of a graph G, $\chi''(G) \leq \Delta(G) + 2$.*

4.1 Definitions

Definition 10. *The **total coloring** of a graph G is a coloring in which no adjacent vertices, no adjacent edges, and no edge and it's two end vertices are assigned the same color. The **total chromatic number**, $\chi''(G)$ of a graph G is the least number of colors needed in any total coloring of G.*

Definition 11. *The **total graph**, $T(G)$ of a graph G is a graph such that*

- *The vertex set of $T(G)$ corresponds to the vertices and edges of G.*
- *Two vertices are adjacent in $T(G)$ if and only if their corresponding elements in G are either adjacent or incident in G.*

Remark 5. *The total coloring becomes a proper vertex coloring of the total graph.*

4.2 Theorems

Theorem 5. $\chi''(G) \geq \Delta(G) + 1$.

Theorem 6. $\chi''(G) \leq \Delta(G) + 10^{26}$.

Theorem 7. $\chi''(G) \leq \Delta(G) + 8\ln^8(\Delta(G))$ *for sufficiently large* $\Delta(G)$.

Theorem 8. $\chi''(G) \leq ch'(G) + 2$, *where ch' is the list edge colorability. It is the least number of k such that G is k-edge-choosable.*

Remark 6. *The conjecture is known to hold for a few important classes of graphs, such as all bipartite graphs and most planar graphs except those with maximum degree 6. The planar case can be completed if Vizing's planar graph conjecture holds, which says, "All simple, planar graphs with maximum degree 6 or 7 are Class 1.". The case where the maximum degree is 7 was solved in 2001; so, all that remains in where the maximum degree is 6 for Vizing's planar graph conjecture.*

5 Uniquely Colorable Polyhedra

Problem 5. *Are the dodecahedron and the tetrahedron the only polyhedra with unique 4-colorings up to symmetry?*

5.1 Background

John Conway notes that the dodecahedron has a unique face 4-coloring up to symmetry. Similarly, the tetrahedron has a unique face 4-coloring up to symmetry. The property is preserved if we (repeatedly) truncate any vertex of the polyhedron of degree 3. Conway asks: *Are these the only polyhedra with unique 4-colorings up to symmetry?*

Theorem 9. *It is known that any cubic polyhedron that is uniquely 4-colorable (not just unique up to symmetry) arises from a truncation of a tetrahedron.*

6 Perfect Edge Coloring the n-cube

Problem 6. *For each integer n there is a coloring of the edges of the n-cube with n colors, one being black, such that the black edges together with the edges of any other color induces a Hamiltonian cycle. [see http://www.emba.uvm.edu/ archdeac/problems/perfectq.htm]*

7 Toroid Graphs

Problem 7. *Every 4-connected toroidal graph is Hamiltonian.*

7.1 Definitions

Definition 12. *A graph G is **toroid** if it can be embedded on the torus.*

Definition 13. *A graph is called k-connected if its vertex connectivity is k or greater. That is to say that a graph G is said to be k-connected if there does not exist a set of $k-1$ vertices whose removal disconnects the graph.*

7.2 Results

I'm a little confused in that it says that Tutte proved that every 4-connected toroidal graph has a hamiltonian cycle, but he conjectured that every 4-connected toroidal graph is hamiltonian. [see http://www.emba.uvm.edu/ archdeac/problems/hamtorus.html]

Theorem 10. *5-connected toroidal graphs are Hamiltonian.*

8 Magic Square

There is no problem for this section. This is more of an open eded problem. We can be a little creative with this one.

Definition 14. *A **magic square** of order n is an arrangement of n^2 numbers, usually distinct integers, in a square, such that the n numbers in all rows, all columns, and both diagonals sum to the same constant. A **normal** magic square contains the integers 1 to n^2.*

Definition 15. *The constant sum in every row, column and diagonal is called the **magic constant**, M. For a normal magic square, that constant is $M = \frac{n(n^2+1}{2}$*

8.1 Variations

- If not only the main diagonals, but also the broken diagonals sum to the magic constant, the result is a **panmagic square**
- If raising each number to certain powers yields another magic square, the result is a **bimagic square** (or similarly, a trimagic or multimagic square)
- We could relax the restriction on the diagonals, making a semimagic square
- We could fill the entries with different numbers (i.e. primes and powers of primes)
- We could alter the shape and perhaps dimension. (e.g. dodecahedron, triangles, stars, hexagons, cubes, tesseracts, hypercubes, etc...)

8.2 Example

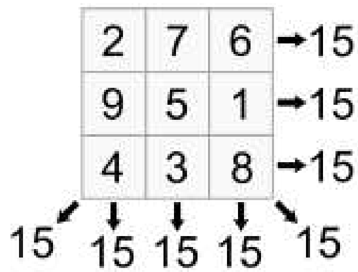

Figure 2: Smallest Known Magic Square

For more information, see Wikipedia.

9 Hadamard Matrices

Problem 8. *A Hadamard matrix of order $4k$ exists for every positive integer k.*

9.1 Definition

Definition 16. *A* ***Hadamard Matrix*** *is a square matrix whose entries are either 1 or -1 and whose rows are mutually orthogonal. in geometric terms, this means that every two distinct rows represent two perpendicular vectors, while in combinatorial terms, it means that every two distinct rows have matching entries in exactly half of their columns and mismatching entries in the remaining columns.*

Remark 7. *It has been problem that the order of a Hadamard matrix must be 1,2, or a multiple of 4; however, it has not been problem that there exists a Hadamard matrix for each of these orders.*

Remark 8. *Sylvester showed a doubling construction for 2,3,8,16,32,etc... Hadamard constructed matrices of order 12 and 20. Raymond Paley discovered a construction that produces a Hadamard matrix of order $q+1$ where $q \cong 3 \mod 4$, producing a Hadamard matrix of order $2(q+1)$, where q is a prime power congruent to $1 \mod 4$.*

10 Latin Squares

Problem 9. *Is there some constant $c > 0$ such that we an always fill at least cn^2 cells given that the Latin square has the Blackburn property.*

Definition 17. *A partial Latin square has the* ***Blackburn property*** *if whenever cells (i, j) and (k, l) are occupied by the same symbol, the opposite corners, (i, l) and (k, j) are empty. What is the highest achievable density of filled cells in a partial Latin square with the Blackburn property? In particular, is there some constant $c > 0$ such that we can always fill at least cn^2 cells?*